# The Mechanical Energy Theorem

The following is a review, for introductory mechanics teachers, of a useful corollary of Newton's laws which I will call the Mechanical Energy Theorem (MET). A conscious motivation will be to contrast the MET with the First Law of Thermodynamics (FLT). In the spirit of introductory texts and lectures I may do violence to completeness and strict rigor; I welcome correction and amplification by others. I promise to NEVER use the word "work":

The MET can be applied to the translational motion of the center of mass (CM) of a single particle, a rigid body, a gas, a liquid, or any system of particles or objects. For example, it can be applied to a system defined as the collection: a taxi cab in NY city, a horse in Wyoming, a commuter train in California, a mosquito in the Everglades, an airplane flying over Moscow and the planet Uranus. The system of interest is simply defined by enumerating the particles and/or objects to be included; the rest of the universe constitutes the "environment" for the defined system. Mechanical interaction between the system and the environment is modeled by Newtonian forces represented by vectors. At every instant of time these forces can be summed vectorialy to obtain the "net external force" on the system. (Even though these forces act on different particles and objects, perhaps widely separated, the vector sum is performed in the usual manner, just as if one were combining the forces acting on a single particle.) Internal forces (interactions between system members) need not be included; by Newton's third law they would contribute zero to the force sum. Also, at every instant of time the position of the CM of our system can be calculated and its trajectory in space can be defined.

The MET states that the line integral of the net external force over the space trajectory of the system CM yields a scalar quantity (with sign) which is numerically equal to the change in the quantity .5 M V^2, calculated at the beginning and end of the trajectory (M is the total mass of the system and V is the speed of the CM). Mathematically: the integral of Fnet (dot) dRcm from point 1 to point 2 equals (.5MV^2)2 minus (.5MV^2)1.
This follows simply by integrating Fnet (dot) dRcm = M(dV/dt) (dot) dRcm over the CM trajectory from point 1 to point 2. This theorem may be applied over any part, or all, of the CM trajectory; of course, the calculations are to be performed in an inertial frame. Within Newtonian Mechanics, there is no exception to this theorem. It is conventional to refer to the quantity .5MV^2 as the CM kinetic energy of the system.

If one already knows about energy conservation and the transfer of energy between objects (who doesn't?), it may be tempting to leap to the conclusion that the MET implies a transfer of energy from the agent(s) of Fnet to the system; ie., that these agents are the sources (or sinks) of the system kinetic energy change. This may or may not be so; in either case the MET says nothing on the subject. For example, when an ice skater pushes off from a wall and thereby gains speed, the MET states that the line integral of the force of the wall on the skater is numerically equal to the skater's kinetic energy increase. However, the wall has not given up any energy to the person; the wall has not lost any energy; the source of the skater's kinetic energy increase is body metabolism acting through body muscle forces. The wall provides a "leverage fulcrum" against which these forces can operate; in so doing the wall force takes measure (through its line integral) of the kinetic energy change. In sum, the MET uses the line integral of the wall force as a measurement of the skater's kinetic energy change. It says nothing about the source of this energy. It is the FLT which performs this function. The FLT does something the MET does not; it identifies and classifies the environmental interactions responsible as sources and sinks of a definable system property - the system energy.

(Aside): Note that as regards energy transfer considerations, a useful model might have the skater’s “push” store energy in an elastic deformation of the tree - the relaxing tree would then return that energy to the skater as it pushes on the skater, just as occurs in elastic collisions. But this is here an aside - the MET is NOT concerned with “energy transfers” - it merely asserts a numerical equality.

The Mechanical Energy Theorem (MET) states that, for any system of particles and objects, the line integral of the net external force over the trajectory of the system CM is numerically equal to the change in the CM kinetic energy. It should seem remarkable to the sensitive physicist that the RHS of this equality depends only on the endpoints of the motion; ie., it does not matter how the CM velocity V(t) varies in time, only the end values of V(t) are used.

(Aside): When, as a teacher, you derive the MET you should pause and marvel at what happens when you integrate the RHS of Fnet (dot) dRcm = M(dV/dt) (dot) dRcm. Here V, dV/dt, and Rcm are all changing in time and it looks very much like one needs to know the details of the motion before the RHS can be integrated. However, the integral of the RHS is easily turned into the sum of three terms of the form M*int{Vi * dVi}, where Vi is a cartesian component of the CM velocity. Remarkably, each of these definite integrals is simply the area under a graph of Vi vs. itself! Thus each term is just M*(.5Vi^2)2 minus M*(.5Vi ^2)1; ie., M times the difference in the area of two triangles, no matter how Vi varies in time! One does not need a calculus course to do this, only an appreciation of the definite integral int{y(x)*dx} as the area under a plot of y(x) vs x.

Wouldn't it be cool if the LHS of the MET, int{Fnet (dot) dRcm}, could also be evaluated by looking at only the end points of the trajectory! Indeed, this can be done for a certain, special kind of force. Suppose one of the external forces depends only on the position of the system CM, so that we can write Fnet = F(Rcm) + F_other, and suppose further that the line integral of F(Rcm ) over any trajectory depends only on the coordinates of the end points of that trajectory and not at all on the particular path chosen between those end points. (For reasons which will become apparent, we shall call such a special force a "conservative force".) One can then choose an arbitrary reference point (Rref) and define a scalar field U(R) which assigns to each point in space a number. That number will be minus the line integral of this conservative force from the reference point Rref to the general point R (Note that such a U(R) will be uniquely definable only for a conservative force.) It is then easily shown that the contribution of F(Rcm) to the line integral of Fnet (dot) dRcm is numerically equal to U(R1) minus U(R2). The MET then can be written as: line int{F_other (dot) dRcm} +U(R1) - U(R2) = CMKE_2 - CMKE_1 . This is conventionally written: line int{F_other (dot) dRcm} = CMKE_2 + U(R2) - CMKE_1 - U(R1) , or line int{F_other (dot) dRcm} = ME_2 - ME_1, where ME_i = CMKE_i + U(Ri)

Since each term has units of energy, we may call U(R) the “potential energy function" (or simply the potential energy), and ME the "total mechanical energy". In the general case, one may include in ME a potential energy term Ui(R) for each conservative force in Fnet. In the fortuitous case where all of the forces are conservative, then F_other = zero, and we have: 0 = ME_2 - ME_1, or ME_1 = ME_2, or ME = constant, or CMKE + U1(R) + U2(R)+ ... =constant. So that if only conservative forces contribute to the line integral of Fnet (dot) dRcm, we can assert a conservation statement: The grand sum of the center of mass kinetic energy and all of the potential energy functions is a constant of the motion. For the general case, when F_other is not zero, we may assert: line int{F_other (dot) dRcm} = delta (ME) = ME_2 - ME_1. So that if there are other forces acting, not represented by a U(R) function, their line integral will be numerically equal to the change in the system's total mechanical energy. This last equality is what I (and most everybody) have always taught as the “work - energy theorem” in its most general (translational) form . I propose it be re-named the Mechanical Energy Theorem (MET) and that the LHS be referred to simply as the CM line integral of F_other. This avoids use of the word “work” which has recently been rendered very confusing by rampant uncritical thinking.

It should be apparent that if there are no conservative forces, then F_other = Fnet, ME = CMKE, and we are back to the original statement of the MET: Int{Fnet (dot) dRcm} = delta (CMKE). This will also be our statement of the MET if there are conservative forces, but we do not choose to replace their line integrals with the potential functions.

Note that a potential energy function U(R) is here simply another way of calculating the line integral of a conservative force, so that no more should be read into the Ui(R) than can be read into the line integrals of the forces simply on the basis of their appearance in the MET. Any further interpretation (eg., as the transfer of particular forms of some universally conserved quantity) will have to come from postulates of a wider scope and of a completely different kind, eg., about energy, its various forms and its conservation, such as are made in the first law of thermodynamics (FLT). But even without additional assumptions there lies a rich mathematical treasure in the further study of the scalar potential field U(r) and its relation to the vector force field F(r).

A conservative force was defined as a vector field F(r) whose line integral through space is a function only of the end points and independent of the path chosen between those points. Three adjunct properties follow immediately:
1) Changing the position of the reference point Rref simply adds a constant to U(r) everywhere. Since the MET involves only the delta U(r) of two space points, the theorem is unaffected. IOW U(r) is definable and meaningful only to within an arbitrary additive constant; it is how the value of U(r) varies from point to point which has physical import.
2) The line integral of a conservative force around any closed path (identical start and finish points) is necessarily zero.
3) In a one dimensional space any f(x) is necessarily conservative.

At first blush it would seem that the "constant" frictional force f = mu*N in a one dimensional problem would qualify under # 3) as a conservative force. Take the time to illustrate that this force does not qualify because it is NOT describable as a f(x) since it changes direction so as to always oppose the velocity vector and thus always contribute a negative quantity to the LHS of the MET. Contrast that behavior with the gravitational force mg, which does not change direction with the velocity. Very instructive; do not omit.

After treating these simplest examples you should now consider the next more involved F(x), a linear function. So integrate the Hooke's law spring force to obtain U(x) = int(o=>x){(kx)dx} . The integral to be done here is again the area under a graph of x vs itself! We can do that! U(x) = .5kx^2 . (It should be apparent that x is the elongation of the spring beyond its relaxed length, x is negative for a compression, and x=0 is the chosen reference point Rref for U=0.)

Staying in one dimension and treating a single mass under the influence of a conservative F(x), show that U(x) = -int{F(x)dx} implies that moving from x to x + dx will produce a change in U(x) given by dU(x) = -F(x) dx (extending the upper limit of the integral by dx simply adds -Fdx to the area under the curve, and this is the change in U(x) .) This yields the important result: F(x) = - dU(x)/dx . This is the inverse of the equation defining U(x) and allows one to recover F(x), given U(x). Note, importantly, that the arbitrary additive constant in U(x) does not enter into this calculation of F(x) from U(x). The force F(x) is equal to minus the space rate of change of the potential function U(x), ie.; minus the slope of U(x). We should add this to our above enumerated properties of U(r) as: 4) In a one dimensional space, F(x) = - dU(x)/dx

Now you have the machinery to treat the Hooke's law oscillator using graphs of U(x) and F(x) ; one the area under minus the other; the other the negative of the slope of the first. You can illustrate how the MET (here a conservation statement) predicts the motion, the turning points, etc. You can add a frictional force which causes a "dissipation" of the mechanical energy, and show how the turning points converge to the stable equilibrium point at $x = 0$. It is not forbidden at this point to leap ahead to discussions of temperature rises, heat energy, and the possibility, at least, of a wider energy conservation postulate.

From the Hooke's law oscillator, generalize to other one dimensional potential functions, show the roller coaster analogy, illustrate stable and unstable equilibrium points, escape velocities, etc. - there is no end, even in just one dimension!

The use of potential energy functions will usually be restricted to problems involving a single particle or a single rigid body, because the conservative force must be a function of the CM position only. However, one can sometimes rework the model description so that the mathematics is the same as if the above were true, even though it was not true in the original model. As a very useful example consider the dumbbell system of two masses interacting through a central force F(|R|); R is the vector locating mass 1 from mass 2 and F(|R|) depends only on its magnitude. This could be the model for a variety of physical systems, from a hydrogen molecule to a binary star system. We are interested in the time behavior of the vector R, ie.; the motion of M1 relative to M2, assuming the dumbbell system is isolated from other forces.

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             R2                         R1

M2<-----------------------CM----------------------------------------------->M1
----------------------------------------------------------------------------->
                              R
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The difficulty is that if we define M1 as our system, the force on it depends not only on its location, but also on the location of M2, which will not stand still for us. However: From the definition of the CM, $M1*R1 + M2*R2 = 0$ (Eq #1)

By construction, $R = R1 - R2 = R1*(1 + M1/M2)$ , after using Eq#1. (Eq #2)

Since the CM is an inertial origin,

$M1*R1'' = F(|R|) * R / |R|$ ; ('' = 2nd time derivative) (Eq#3)

Using Eq#2, $m*R'' = F(|R|) * R / |R|$ (Eq #4),

where $m = (M1*M2)/(M1+M2)$, the "reduced" mass.

Eq (4) describes the behavior of the vector R, which locates M1 from (the moving) M2, and says that its behavior is the same as that of a particle of mass m under the influence of the central force F(|R|) of a FIXED source. Obviously, the MET can use a potential energy function to describe the behavior of R in Eq #4.

We next develop the general, three dimensional counterparts of the one dimensional properties of the potential function U(r) discussed earlier. I have tried to make things transparent enough for an introductory course, but each teacher will have to judge for herself just how far time and reality permit her to go in each class (some of the language is perforce loose and does violence to complete rigor).

We have seen that in one dimension, any honest to goodness F(x) is necessarily conservative. In three dimensions, the necessary and sufficient condition for F(r) to be conservative is that the magnitude of F(r) cannot vary in value as one moves infinitesimally in a direction perpendicular to the direction of the vector F(r). More precisely, if the direction of F(r) at a space point is defined to be the x direction, then at that point $dF(x,y,z)/dy = 0$ , and $dF(x,y,z)/dz = 0$ (partial derivatives). The drawing below will show that if this condition is not satisfied, you can do the line integral around a closed infinitesimal path and get a non-zero result:

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      F(x,y+dy,z)
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      F(x,y,z)
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If  F(r) behaves as shown (F points in the x direction, y is up the page), the line integral counterclockwise around the square (dx by dy)  would yield F(x,y,z)*dx - F(x,y+dy,z)*dx .  This is zero only if F(x,y+dy,z) = F(x,y,z), or equivalently, the partial derivative dF(x,y,z)/dy = 0.  In the same way, dF(x,y,z)/dz =0 is required for a conservative force.

In mathematical language, this requirement is worded: Curl F(r) =0.  This makes clear the physical requirement on a conservative force, without going into the full, general machinery which generates the (vector) Curl F(r) . The helpful paddle wheel picture of the curl might be introduced here; ie.,  if F(r) in the above drawing is taken to represent the water velocity in a stream, a paddle wheel introduced into the stream with its axle pointing out of the page will not rotate if Curl F(r)  = 0.  Not surprisingly, a vector field with zero curl is often called an irrotational field.  (Advanced students would hear about Stokes' theorem at this point.)

To get the general, three dimensional, relation which recovers F(r) given U(r), observe that a general vector displacement dr from some space point r will encounter a change dU(r) in the scalar U(r).  From U(r) = - int{F(r) (dot) dr} we have dU(r) = -F(r) (dot) dr, for any vector "step" dr.  Now if dr is chosen to be completely in the x direction, we have dU(r) = -F_x * dx. Similarly for steps in the y, z directions the changes in U(r) are given by dU(r) = -F_y  * dy and dU(r)= -F_z * dz, respectively.

These statements are equivalent to F_x = -dU(x,y,z)/dx ;  F_y = -dU(x,y,z)/dy ;  F_z = -dU(x,y,z)/dz  (partial derivatives). Conventional notation combines these three component statements into the single vector statement F(r) = -Grad{U(r)},  "F of r equals minus the gradient of U of r".  An equivalent statement is that at any space point r the component of the conservative force F(r) in any chosen direction is equal to minus the space rate of change of its potential function U(r) in that direction, or symbolically F_s = -dU(r)/ds for any direction s. The F(r) <==> U(r) mathematical machinery which we have developed has analogous applications in several other fields of physics and engineering.  For example, if the scalar U(r) gives the temperature at each space point r, there is a heat conduction equation, analogous to F(r) = - Grad{U(r)}, which says that thermal energy will "flow" in the direction -Grad{U(r)}, ie.; from positions of higher temperature to positions of lower temperature.  Students can more easily visualize this thermal situation and can then be led to the "picture" of a particle subjected to the force F(r) accelerating toward lower values of the potential function U(r).

Addendum:

I have avoided using the word "work", not to champion any crusade to abolish use of that word, but simply to avoid the senseless semantic arguments which inevitably occur whenever one asks, or answers, the questions "Does such and such a force do “work” in such and such a situation?" , “But how can an agent’s force do work and yet not transfer energy?”, etc. The physics can be unambiguously expressed, and such arguments avoided, by simply avoiding use of that multi-valued word.

Let me close by enumerating a few textbook problems which traditionally apply the MET assertion that KE + U(r) = A Constant of the Motion  when Fnet = -GradU(r):

1) Near earth free fall, where F =  -mg = -Grad (mgy) ==>  KE +mgy = Constant
2) The planet problem, where F = -GMm/r^2 = -Grad (-GMm/r) ==> KE - GMm/r = Constant
3) Simple Harmonic Motion, where F = -kr = -Grad (.5kr^2) ==> KE + .5kr^2 = Constant

Note that, within Newtonian Dynamics, these “Constant of the Motion” assertions [KE + U(r) = Constant] are not “Conservation of Energy” assertions in the First Law of Thermodynamics sense - they are simply applying the MET (aka the Work-Energy theorem), which depends ONLY on Newton’s laws of motion, and knows nothing of the “Work” and “Energy” concepts of the First Law of Thermodynamics.  The U(r) function of the MET is NOT proposed as an FLT “Energy” - it is simply a scalar function of position whose negative gradient  numerically equals a zero curl force.  The endless “reification” haggles over “Where is the potential energy?” and “To what object(s) does the potential energy belong?”, etc have no relevance to the MET’s U(r).

The MET simply quantifies just how the forces on a system change the kinetic energy of that system - and, for zero curl forces the MET uncovers a constant of the motion.

Thank you if you have read this far.  I hope that there is some help here for someone.  Ciao!

-Bob Sciamanda

La Comedia e finita!
- I Pagliacci" (ending words), by Ruggiero Leoncavallo