Constructing a well tempered scale

To guarantee simple transposition we require $f_{n+1} / f_n = g$, a constant. **Then $f_n/f_0 = g^n$.**
With N notes per octave, we must require $f_N / f_0 = g^N = 2.0$
Then g must be the Nth root of 2.0 ie., $g = 2^{(1/N)}$, so that **$f_n / f_0 = 2^{(n/N)}$**

<u>How to choose N, the number of notes per octave?</u>

We will choose that N which includes notes having f_n / f_0 ratios close to all these desirable (Pythagorean) values ==>

Third = 5/4=1.25; Fourth = 4/3=1.33; Fifth = 3/2=1.5; Sixth=5/3=1.66; Octave = 2.0

$$n := 1, 2 .. 20 \qquad g(N,n) := 2^{n \div N}$$

$$g(6,n)=\begin{bmatrix}1.122\\1.26\\1.414\\1.587\\1.782\\2\\\vdots\end{bmatrix} \quad g(7,n)=\begin{bmatrix}1.104\\1.219\\1.346\\1.486\\1.641\\1.811\\2\\\vdots\end{bmatrix} \quad g(8,n)=\begin{bmatrix}1.091\\1.189\\1.297\\1.414\\1.542\\1.682\\1.834\\2\\\vdots\end{bmatrix} \quad g(9,n)=\begin{bmatrix}1.08\\1.167\\1.26\\1.361\\1.47\\1.587\\1.714\\1.852\\2\\\vdots\end{bmatrix}$$

$$g(10,n)=\begin{bmatrix}1.072\\1.149\\1.231\\1.32\\1.414\\1.516\\1.625\\1.741\\1.866\\2\\\vdots\end{bmatrix} \quad g(11,n)=\begin{bmatrix}1.065\\1.134\\1.208\\1.287\\1.37\\1.459\\1.554\\1.656\\1.763\\1.878\\2\\\vdots\end{bmatrix} \quad g(12,n)=\begin{bmatrix}1.059\\1.122\\1.189\\1.26\\1.335\\1.414\\1.498\\1.587\\1.682\\1.782\\1.888\\2\\\vdots\end{bmatrix} \quad g(13,n)=\begin{bmatrix}1.055\\1.113\\1.173\\1.238\\1.306\\1.377\\1.452\\1.532\\1.616\\1.704\\1.798\\1.896\\2\\\vdots\end{bmatrix}$$

$$g(14,n)=\begin{bmatrix}1.051\\1.104\\1.16\\1.219\\1.281\\1.346\\1.414\\1.486\\1.561\\1.641\\1.724\\1.811\\1.903\\2\\\vdots\end{bmatrix}\quad g(15,n)=\begin{bmatrix}1.047\\1.097\\1.149\\1.203\\1.26\\1.32\\1.382\\1.447\\1.516\\1.587\\1.662\\1.741\\1.823\\1.91\\2\\\vdots\end{bmatrix}\quad g(16,n)=\begin{bmatrix}1.044\\1.091\\1.139\\1.189\\1.242\\1.297\\1.354\\1.414\\1.477\\1.542\\1.61\\1.682\\1.756\\1.834\\1.915\\2\\\vdots\end{bmatrix}\quad g(17,n)=\begin{bmatrix}1.042\\1.085\\1.13\\1.177\\1.226\\1.277\\1.33\\1.386\\1.443\\1.503\\1.566\\1.631\\1.699\\1.77\\1.843\\1.92\\2\\\vdots\end{bmatrix}$$

$$g(18,n)=\begin{bmatrix}1.039\\1.08\\1.122\\1.167\\1.212\\1.26\\1.309\\1.361\\1.414\\1.47\\1.527\\1.587\\1.65\\1.714\\1.782\\1.852\\1.924\\2\\\vdots\end{bmatrix}\quad g(19,n)=\begin{bmatrix}1.037\\1.076\\1.116\\1.157\\1.2\\1.245\\1.291\\1.339\\1.389\\1.44\\1.494\\1.549\\1.607\\1.667\\1.728\\1.793\\1.859\\1.928\\2\\\vdots\end{bmatrix}\quad g(20,n)=\begin{bmatrix}1.035\\1.072\\1.11\\1.149\\1.189\\1.231\\1.275\\1.32\\1.366\\1.414\\1.464\\1.516\\1.569\\1.625\\1.682\\1.741\\1.803\\1.866\\1.932\\2\end{bmatrix}$$

N = 12 or 19 notes per octave will do !

Note that N = 7 is also a pretty good fit, but it would include only the 7 notes Doe, Re, Mi, Fa, So, La, Ti and so would be lacking in variety. Thus the 12 note scale is chosen because the 7 note scale is too sparse and the 19 note scale is too dense.