The Pythagorean scale factors

$n := 0, 1 .. 15$ $\quad f(n) := \left(\frac{3}{2}\right)^n$ $\quad g(n) := \frac{f(n)}{2}$ $\quad h(n) := \frac{f(n)}{4}$ $\quad j(n) := \frac{f(n)}{8}$

$n =$	$f(n) =$	$g(n) =$	$h(n) =$	$j(n) =$
0	1	0.5	0.25	0.125
1	1.5	0.75	0.375	0.188
2	2.25	1.125	0.563	0.281
3	3.375	1.688	0.844	0.422
4	5.063	2.531	1.266	0.633
5	7.594	3.797	1.898	0.949
6	11.391	5.695	2.848	1.424
7	17.086	8.543	4.271	2.136
8	25.629	12.814	6.407	3.204
9	38.443	19.222	9.611	4.805
10	57.665	28.833	14.416	7.208
11	86.498	43.249	21.624	10.812
12	129.746	64.873	32.437	16.218
13	194.62	97.31	48.655	24.327
14	291.929	145.965	72.982	36.491
15	437.894	218.947	109.473	54.737

Note: If you were to repeat this scheme, but begin on a different f(n) as your f(0), you would get a completely different set of notes! This scale will not transpose. It generates a unique "key".

Note that octave notes would occur at

f(n)=2^n ---->

$$2^n = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \\ 64 \\ 128 \\ 256 \\ 512 \\ 1.024 \cdot 10^3 \\ 2.048 \cdot 10^3 \\ \vdots \end{bmatrix}$$

This means that an octave note occurs when (3/2)^n = 2^m for some integers n & m. This NEVER happens - there is no exact solution. Octave notes are NOT included.

Many compromise fixes have been proposed and used ===>

Example compromise fix (the "Meantone" scale):

Tweak the value 3/2 a little to a new value r so that there exist m and n integers such that:

r^n = 2^m . Equivalently, m = n Log(r)/Log(2)

If r were exactly 3/2 , this would give m = n*0.58496 .

Try n=1,2,3... until m ~ an integer -->

$$m(n) := n \cdot \left(\frac{\ln(1.5)}{\ln(2)}\right) \quad \text{--->}$$

$$m(n) = \begin{bmatrix} 0 \\ 0.585 \\ 1.17 \\ 1.755 \\ 2.34 \\ 2.925 \\ 3.51 \\ 4.095 \\ 4.68 \\ 5.265 \\ 5.85 \\ 6.435 \\ 7.02 \\ 7.605 \\ 8.189 \\ 8.774 \end{bmatrix}$$

n=12 gives m = 7.02. So, using n=12 & m=7 . Then -->

r=2^(m/n) gives r=1.498307 as the compromise to 3/2.

Thus, using f(n)= r^n, with r=1.498307, we will get an octave at n=12, giving a scale of 12 intervals per octave. This is a "well tempered" scale, but this scale won't transpose. It generates a single unique key.

But, importantly, this exercise shows that 12 notes per octave would be a serendipitous choice!

The "even tempered" scale

Guarantee transposability by imposing a constant frequency ratio of adjacent notes:

f(n+1)/f(n) = a constant, for all n. For a scale of 12 notes per octave this means-->

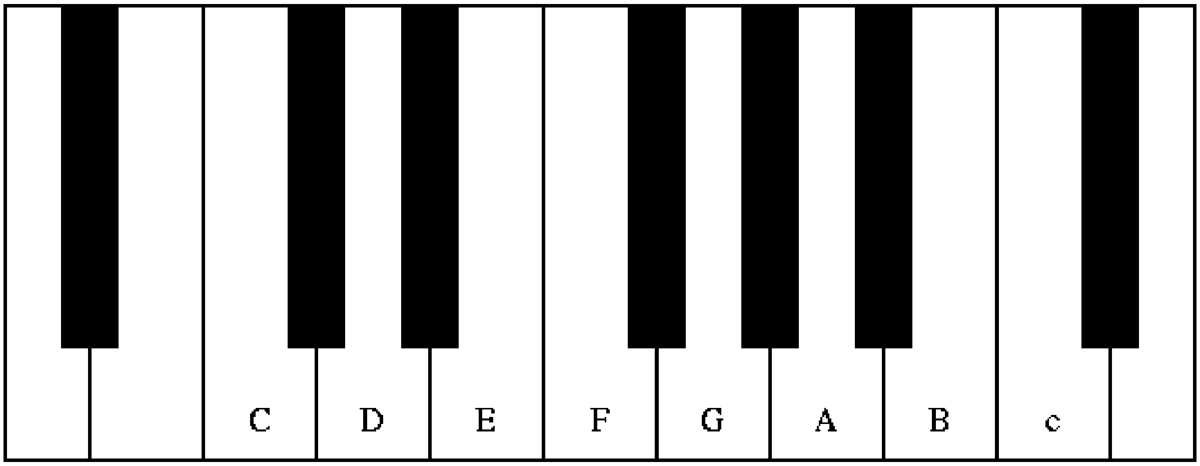
f(n+1) =2^(1/12)* f(n) or f(n) = 2^(n/12)f(0)

$$2^{\frac{n}{12}} = \begin{bmatrix} 1 \\ 1.059 \\ 1.122 \\ 1.189 \\ 1.26 \\ 1.335 \\ 1.414 \\ 1.498 \\ 1.587 \\ 1.682 \\ 1.782 \\ 1.888 \\ 2 \\ 2.119 \\ 2.245 \\ 2.378 \end{bmatrix}$$

Note that the "fifth" ratio of 3/2 is well approximated by f(7)=1.498 *f(0). Because of the constant ratio scheme, EVERY note has such a "fifth" companion note exactly 7 intervals away.

Note further that the five particularly important intervals are all well approximated ==> the octave (2/1)=2.0, a just fifth (3/2)=1.5, a just fourth (4/3)=1.33, a just major third (5/4)=1.25 and a just major sixth (5/3)=1.67.

Finally, note that generating a second piano of notes by using any other already established note as your f(0) will repeat the identically same piano of notes.



Consonant f/f_0 ratios in 12 note octave

Modern Names		Just	Even Temp ($2^{n/12} * f_0$)
C ==> E	THIRD	5/4 = 1.25	1.26
C ==> F	FOURTH	4/3 = 1.33 . . .	1.335
C ==> G	FIFTH	3/2 = 1.5	1.498
C ==> A	SIXTH	5/3 = 1.66 . . .	1.682
C ==> C'	OCTAVE	2/1 = 2.0	2.0

The 2^1/12 ratio (guarantees transposability) adds the black key notes (and D & B).